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ON SINGULARITIES IN THE SOLUTION OF A PROBLEM FOR AN ELASTIC HALF-PLANE WITH A ROD EMERGING ORTHOGONALLY ON THE BOUNDARY*

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An integro-differential equation in the contact stress is investigated for the problem for an elastic half-plane (isotropic and orthotropic) with an extensible rod emerging orthogonally on the boundary. The question of the nature of the contact stress singularity at the point where the rod emerges at the boundary is clarified. It is shown that the asymptotic form of the Cauchy-type integral does not enable a single-valued solution of the question of the nature of the singularity to be obtained if it is assumed to be power-law logarithmic. The true nature of the singularity is determined by an exact solution of the equation constructed using the Mellin integral transform and the Carleman boundary value problem for a strip /1/.

The problem of the contact of an elastic half-plane with an extensible rod emerging orthogonally at the boundary was examined in /2/ in the isotropic case. A deduction is made there about the presence of a power-law singularity in the contact stresses at the point where the rod emerges at the boundary. This same problem was examined in the orthotropic case in /3/, in which a deduction is made on the basis of an asymptotic investigation and an exact solution of a problem constructed in the limit case of strong orthotropy that the contact stress singularity is of a power-law logarithmic nature, unlike /2/, where it becomes logarithmic for zero values of Poisson's ratio. In /3/ doubts were raised about the validity of the deductions made earlier in /2/ on the nature of the singularity, and the validity of taking a power-law logarithmic singularity for the isotropic case also.

It is shown below that the asymptotic form of a Cauchy-type integral does not enable the question of the nature of the singularity to be solved uniquely, if it is assumed to be power-law logarithmic. Consequently, the validity of the deductions in /2/ is confirmed; the singularity also turns out to be a power-law for an orthotropic half-plane and goes over into the logarithmic law only in the case of limiting strong orthotropy, which refutes the corresponding deduction in /3/.

1. The problem for an elastic half-plane $(-\infty < x < \infty,\ y > 0)$ with a rod extensible by a force Q and located on a line $(x = 0,\ 0 < y < l)$ is reduced in /2, 3/ to an integro-differential equation in the axial force in the rod $\varphi_n(y)$

$$\varphi_n(y) + \mu_n \int_0^l K_n(y, s)\, \varphi_n'(s)\, ds = 0, \quad 0 < y < l, \quad n = 1, 2 \tag{1.1}$$

$$\varphi_n(0) = Q, \quad \mu_1 = SE_0 (4\pi hE)^{-1}, \quad \mu_2 = SE_0 (1 - \nu_1\nu_2)(2\pi hE_1)^{-1}$$

$$K_1(y, s) = \frac{(3-\nu)(1+\nu)}{y-s} + \frac{8-(3-\nu)(1+\nu)}{y+s} + \frac{2(1+\nu)^2 s(y-s)}{(y+s)^3}$$

$$K_2(y, s) = \left(\frac{A_1}{\lambda_1} - \frac{A_2}{\lambda_2}\right)\frac{1}{y-s} + \left(\frac{A_3}{\lambda_1} + \frac{A_4}{\lambda_2}\right)\frac{1}{y+s} -$$
$$- \frac{A_5}{\lambda_1 y + \lambda_2 s} - \frac{A_6}{\lambda_2 y + \lambda_1 s}$$

The value $n = 1$ characterizes the isotropic case, $n = 2$ the orthotropic case, S and E_0 are the cross-sectional area and the elastic modulus of the rod, h is the plate thickness, E, ν and $E_{1,2}$, $\nu_{1,2}$ are the elastic moduli and Poisson's ratios, respectively, in the isotropic and orthotropic cases, and the constants $A_j\ (j = 1, 2, \ldots, 6)$ and $\lambda_{1,2}$ are related to the elastic properties of the orthotropic material and are determined in /3/. The function $\varphi_n'(s)$ is sought in the class of integrable functions.

Besides the Cauchy kernel, the kernels of the equation contain components having a fixed first-order singularity at the point $y = 0$. The presence of these components changes the

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nature of the behaviour of the function $\varphi_n'(s)$ as $s \to 0$, it ceases to be a radical as, for instance, as $s \to l$, and requires refinement. Knowledge of the nature of the singularity of the solution is important to be able to construct effective approximate solutions, where the need for their construction is dictated either by there not being an exact solution of the problem, or by the complexity of its numerical realization.

First we will refine the nature of the singularity of the function $\varphi_n'(s)$ as $s \to 0$, being guided by the scheme in /2, 3/, which is based on the asymptotic form of a Cauchy-type integral near the neds of the line of integration, and by setting $l = \infty$ for simplicity. We will show that this scheme does not resolve the question of the nature of the singularity uniquely, if it is assumed to be a power-law logarithmic singularity.

Indeed, let the function $\varphi_n(y) \in C_1(0, \infty)$ and a function $\psi_n(s) \in H[0, \infty]$ exist such that

$$\varphi_n'(s) = \psi_n(s)\, s^{-\gamma_n} \ln^k s,\ 0 < \operatorname{Re}\gamma_n < 1,\ k \text{ is an integer} \tag{1.2}$$

The representation (1.2) generalizes the assumptions in /2, 3/ about the nature of the singularity $\varphi_n'(s)$ in which we set $k = 0$ and $k = 1$, respectively. To determine the number γ_n we will write down the asymptotic form of the singular integrals in (1.1) by taking the asymptotic form of an integral of Cauchy type with density (1.2) in the neighbourhood of the point $y = 0$ (/4/, Ch.1, Sect.8.6) as basis:

$$\int_0^\infty \varphi_n'(s) \frac{ds}{s-y} = -\frac{\pi}{\operatorname{tg}\pi(1-\gamma_n)} \psi_n(0)\, y^{-\gamma_n} \ln^k y + \varphi_1(y) \tag{1.3}$$

$$\int_0^\infty \varphi_n'(s) \frac{ds}{s+ay} = \frac{\pi}{\sin\pi(1-\gamma_n)} \psi_n(0)\,(ay)^{-\gamma_n} \ln^k y + \varphi_2(y)$$

$$\int_0^\infty \varphi_n'(s) \frac{s(y-s)\,ds}{(y+s)^3} =$$

$$-(1-\gamma_n)^2 \frac{\pi}{\sin\pi(1-\gamma_n)} \psi_n(0)\, y^{-\gamma_n} \ln^k y + \varphi_3(y)$$

$$a = \text{const} > 0,\quad \varphi_m(y) = O(y^{-\gamma_n} \ln^{k-1} y),\ y \to 0,\ m = 1, 2, 3$$

The requirement that (1.1) should be valid even at the point $y = 0$ taking the asymptotic form (1.3) into account, enables us, after multiplying both sides of the equation by $y^{\gamma_n} \ln^{-k} y$ and passing to the limit as $y \to 0$, to write down the characteristic equations to determine the constants γ_n:

$$f_1(\gamma_1) = 0,\ f_2(\gamma_2) = 0 \tag{1.4}$$

$$f_1(\gamma) = \cos\pi(1-\gamma) - 2\varkappa^{-1}(1-\gamma)^2 + (\varkappa^2+1)(2\varkappa)^{-1},$$

$$\varkappa = (3-\nu)(1+\nu)^{-1}$$

$$f_2(\gamma) = \cos\pi(1-\gamma) - \frac{A_5\lambda_2}{A_1\lambda_2 - A_3\lambda_1}\left[\left(\frac{\lambda_1}{\lambda_2}\right)^{1-\gamma} + \left(\frac{\lambda_2}{\lambda_1}\right)^{1-\gamma}\right] + \frac{A_3\lambda_2 + A_4\lambda_1}{A_1\lambda_2 - A_2\lambda_1}$$

The first equation corresponds to the isotropic, and the second to the orthotropic cases. According to /2, 3/, both equations have no complex roots whose real parts lie in the interval (0,1) under the conditions $0 < \nu < 1/_2$ and $\lambda_1^2 > 0$, respectively. Under these same conditions, both equations have just one real root $\gamma_{1,2} \in (0, 1)$, while $\gamma_1 = \gamma_2 = 0$, respectively, under the conditions $\nu = 0$ and $\nu_1 = \nu_2 = 0$.

Therefore, under the above-mentioned conditions for an isotropic and orthotropic material, the singularity of the function $\varphi_n'(s)$ at the edge of the rod has the form

$$\varphi_n'(s) = O(s^{-\gamma_n} \ln^k s),\ s \to 0$$

We should set $\gamma_1 = \gamma_2 = 0$ in the case $\nu = 0$ and $\nu_1 = \nu_2 = 0$ and the singularity will become logarithmic.

The results obtained, as well as the results in /2, 3/, are not in mutual agreement. This is explained by the fact that the asymptotic of the Cauchy type integral (1.3) enables us to determine just the constants γ_n without in any way fixing the number k, assumed here to be an integer to simplify the discussion. Rejection of this assumption does not cause any difficulties in principle since the formulas for non-integer k, analogous to (1.3), can be obtained by a well-known scheme (/5/, Sect.6.7).

The above result enables us to deduce that the conclusion in /3/ about the falseness of the assumption in /2/, associated with the determination of the nature of the singularity in the function $\varphi_n'(s)$ is without foundation. This is verified below by separating the singularity in the function $\varphi_n'(s)$ on the basis of the exact solution of (1.1).

2. Let us construct the exact solution of (1.1). To this end, we convert it to the form

$$\varphi_n(y) + \mu_n \int_0^\infty g_n\left(\frac{y}{s}\right) \varphi_n'(s) \frac{ds}{s} = 0,\quad y > 0 \tag{2.1}$$

$$g_1(x) = \frac{(3-\nu)(1+\nu)}{x-1} + \frac{8-(3-\nu)(1+\nu)}{x+1} + \frac{2(1+\nu)^2}{(x+1)^2} - \frac{4(1+\nu)^2}{(x+1)^3}$$

$$g_2(x) = \left(\frac{A_1}{\lambda_1} - \frac{A_2}{\lambda_2}\right)\frac{1}{x-1} + \left(\frac{A_3}{\lambda_1} + \frac{A_4}{\lambda_2}\right)\frac{1}{x+1} - \frac{A_5}{\lambda_2}\frac{1}{\lambda_1 x\lambda_2^{-1}+1} - \frac{A_6}{\lambda_1}\frac{1}{\lambda_2 x\lambda_1^{-1}+1}$$

The integral term in (2.1) is a Mellin convolution for the transformation of the same name /6/

$$\Phi(p) = \int_0^\infty \varphi'(s)s^{p-1}ds, \quad \varphi'(s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Phi(p)s^{-p}dp \tag{2.2}$$

We shall seek the solution of (2.1) in the class of functions $\varphi_n'(s)$ possessing the asymptotic $\varphi_n'(s) = O(s^{-\varepsilon})$, $s \to 0$, $0 < \varepsilon < 1$; $\varphi_n'(s) = O(s^{-1-\delta})$, $s \to \infty$, $\delta \geqslant 1$. In this case, for the Mellin integrals to exist, a constant c in the integrals (2.2) should be selected from the interval $(\varepsilon, 1)$ from the functions in (2.1) (in practice, the contour of integration in the second of the integrals (2.2) should be to the left of the line $\operatorname{Re} p = 1$).

We apply the Mellin transform (2.2) to equation (2.1). Using its properties /6/, we arrive at a Carleman boundary value problem for the strip /1/

$$\Phi_n(p_0+1) + \mu_n G_n(p_0)\Phi_n(p_0) = 0, \ \operatorname{Re} p_0 = c \tag{2.3}$$
$$G_1(p) = -16\pi\varkappa(\varkappa+1)^2 f_1(p)[\sin\pi(1-p)]^{-1}$$
$$G_2(p) = -\pi(A_1\lambda_2 - A_2\lambda_1)(\lambda_1\lambda_2)^{-1}f_2(p)[\sin\pi(1-p)]^{-1}$$

The function $f_n(p)$ is defined by (1.4), and the function $\Phi_n(p)$ is analytic in the strip $c < \operatorname{Re} p < 1 + c$. The method of partial factorization /1/ enables us to reduce problem (2.3) to the standard form

$$\Psi_n(p_0+1) + K_n(p_0)\Psi_n(p_0) = 0, \ \operatorname{Re} p_0 = c \tag{2.4}$$
$$\Psi_n(p) = \Phi_n(p)\theta_n^{-p}[\Gamma(p)\cos(^1/_2)\pi p]^{-1}$$
$$\theta_1 = 16\pi\mu_1\varkappa(\varkappa+1)^{-2}, \ \theta_2 = \pi\mu_2(A_1\lambda_2 - A_2\lambda_1)(\lambda_1\lambda_2)^{-1}$$
$$K_n(p) = \operatorname{ctg}(^1/_2\pi p)f_n(p)[\sin\pi(1-p)]^{-1}$$

Here the function $\Psi_n(p)$ is analytic in the strip $c < \operatorname{Re} p < 1 + c$ everywhere with the exception of the point $p = 1$, where a simple pole is found. The function $K_n(p)$ is continuous on the line $\operatorname{Re} p = c$, does not vanish, possesses an asymptotic form $K_n(p) = 1 + O(\exp(-^1/_2\pi|p|))$, $|p| \to \infty$, and has an increment of the argument equal to zero. In this case the solution of the Carleman boundary value problem for the strip (2.4) is given by the formulas /1/:

$$\Psi_n(p) = B_n X_n(p)[\sin\pi p]^{-1}, \ c < \operatorname{Re} p < 1 + c \tag{2.5}$$

$$X_n(p) = \exp\left\{\int_{c-i\infty}^{c+i\infty}\ln K_n(s)[e^{2\pi i(s-p)}-1]^{-1}ds\right\}$$

Here B_n is an arbitrary constant, which sould be fixed by the condition $\varphi_n(0) = Q$. It is convenient to realize it in the formula $\Phi_n(1) = -Q$, which is equivalent to the initial condition since $\varphi_n(\infty) = 0$. As a result of realization of the transformed condition, we obtain, using (2.4) and (2.5),

$$B_n = -2Q[\theta_n X_n(1)]^{-1} \tag{2.6}$$

The second formula in (2.2), as well as (2.4)-(2.6), yield a solution of the integral equation (1.1) in the form of the Mellin integral

$$\varphi_n'(y) = -\frac{Q}{\theta_n}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(p)X_n^+(p)}{X_n(1)\sin(^1/_2\pi p)}\left(\frac{y}{\theta_n}\right)^{-p}dp \tag{2.7}$$

Here $X_n^+(p)$ is the limit value of the function $X_n(p)$ that is analytic in the strip $c < \operatorname{Re} p < c + 1$ to the right of the line $\operatorname{Re} p = c$. To clarify the behaviour of the function $\varphi_n'(y)$ as $y \to 0$, the value of $X_n^+(p)$ should first be replaced in the integral (2.7) by the limit value of the function $X_\pi^-(p)$ that is analytic in the strip $-1 + c < \operatorname{Re} p < c$ to the left of the line $\operatorname{Re} p = c$. The replacement is made by means of the formula /1/

$$X_n^-(p) = K_n(p)X_n^+(p), \ \operatorname{Re} p = c$$

After this replacement, the integrand in the integral (2.7) is analytic everywhere in the strip $-1 + c < \operatorname{Re} p < c$, with the exception of the zeros of the function $f_n(p)$. Applying the theorem on residues to this strip, we replace the integral over the line $\operatorname{Re} p = c$ by the sum of residues relative to the zero of the function $f_n(p)$ and the integral over the line $\operatorname{Re} p = -1 + c$. This residue indeed determines the behaviour of the function $\varphi_n'(y)$ on the edge of the rod.

Taking into account the arrangement of the simple roots of (1.4), we find that for $0 < \nu < {}^1/_2$ and $\lambda_1^2 > 0$, respectivley, the function $\varphi_n'(y)$ has a power-law singularity of the form $\varphi_n'(y) = O(y^{-\gamma_n})$, where γ_n are real zeros of the function $f_n(p)$. For $\nu = 0$ and $\nu_1 = \nu_2 = 0$ we have $\gamma_1 = 0$ and $\gamma_2 = 0$, therefore, the function $\varphi_n'(y)$ is here bounded as $y \to 0$.

To determine the behaviour of the function $\varphi_n'(y)$ at infinity, Cauchy's theorem should be applied to the integral (2.7) and the contour of integration should be shifted from the line $\operatorname{Re} p = c$ to the line $\operatorname{Re} p = 1 + c$. Then utilizing the formula connecting the limit values of the function $X_n(p)$ on the line $\operatorname{Re} p = 1 + c$, we find, by the theorem on residues, that $\varphi_n'(y) = O(y^{-2})$. This confirms that the class in which the solution of (2.1) is sought has been selected correctly $(\delta = 1)$.

Let us examine the limit case of strong orthotropy $(\varepsilon \to 0)$, for which the exact solution has been constructed in /3/, and the logarithmic singularity of the function $\varphi_2'(y)$ as $y \to 0$ has been clarified on the basis of this. In this case the kernel of (1.1) takes the form /3/ (G_0 is the shear modulus of the orthotropic material)

$$K_2(y, s) = \sqrt{\frac{E_1}{G_0}} \left(\frac{1}{y-s} + \frac{1}{y+s}\right) \tag{2.8}$$

Repeating the calculations made above, we arrive at the Carleman problem (2.3) with the coefficient

$$G_2(p) = -\pi\sqrt{E_1 G_0^{-1}} f_2(p) [\sin \pi (1-p)]^{-1}, \; f_2(p) = \cos \pi (1-p) + 1$$

which is factorized into elementary functions and enables us to put $X(p) \equiv 1$ in (2.5)–(2.7) and to write the solution of (1.1) and (2.8) thus:

$$\varphi_2'(y) = -\frac{Q}{\theta_2} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\theta_2^p \Gamma(p)}{\sin({}^1/_2\pi p)} y^{-p}\, dp, \quad \theta_2 = \frac{SE_0}{h\sqrt{E_1 G_0}} \tag{2.9}$$

The behaviour of the function (2.9) as $y \to 0$ is determined by the nearest singular point of the integrand to the line $\operatorname{Re} p = c$ in the half-plane $\operatorname{Re} p < c$. Such a point is the second-order pole $p = 0$. The residue relative to this pole has a component containing the factor $\ln y$. Therefore, the function $\varphi_2'(y)$ defined by the integral (2.9) has a logarithmic singularity as $y \to 0$, which agrees with the result obtained in /3/ for (1.1) and (2.8).

Therefore, the dependence of the singularity of the function $\varphi_2'(y)$ on the edge of the rod on the relationships between the elastic constants has been found indirectly by locating the zeros of the function $f_2(p)$ in the strip $0 < \operatorname{Re} p < 1$. Thus, for $\lambda_1^2 > 0$ the function $f_2(p)$ has a ingle real simple zero $\gamma_2 > 0$ that determines the asymptotic $\varphi_2'(y) = O(y^{-\gamma_2})$, $y \to 0$. For $\nu_1 = \nu_2 = 0$ the zero γ_2 is at the point $p = 0$ and the function $\varphi_2'(y)$ becomes bounded as $y \to 0$, while for $\nu_1 = \nu_2 = 0$ and the limit case of strong orthotropy, the point $p = 0$ becomes a zero of multiplicity two for the function $f_2(p)$, which determines the asymptotic form $\varphi_2'(y) = O(\ln y)$, $y \to 0$.

Thus, the assumption /2/ of a power-law form of the contact stress singularity on the edge of a rod is completely justified, while the assumption in /3/ of a power-law logarithmic form for this singularity for $\lambda_1^2 > 0$ and a logarithmic-type form for $\nu_1 = \nu_2 = 0$ is incorrect.

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REFERENCES

1. REUT V.V. and TIKHONENKO L.YA. Bending of wedgelike plates with elastically fastened or reinforced faces, PMM, Vol.44, No.1, 1980.
2. MUKI R. and STERNBERG E., On the diffusion of load from a transverse tension bar into a semi-infinite elastic sheet, Trans. ASME, Ser. E. J. Appl. Mech., Vol.35, No.4, 1968.
3. MANEVICH L.I. and PAVLENKO A.V., On the nature of the singularity of contact stresses in load transfer from a rod to a semi-infinite elastic plate. Urgent Problems of the Mechanics of Deformable Media, Izdatel'stvo Dnepropetrovsk. Univ. 1979.
4. GAKHOV F.D., Boundary Value Problems. Nauka, Moscow, 1977.
5. CHIBRIKOVA L.I., Fundamental Boundary Value Problems for Analytic Functions. Izdatel'stvo Kazan' Univ., Kazan', 1977.
6. TITCHMARSH E., Introduction to the Theory of Fourier Integrals /Russian translation/, Gostekhizdat, Moscow-Leningrad, 1948.

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